Frequency Effects of Electromagnetic Scattering from Underdense Turbulent Plasmas

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Generalized relations are presented for the first and second moments of the scattered power spectrum function, for underdense turbulent plasmas, in terms of the spatial distribution of electron density and velocity fluctuations, and of mean velocity. The scattering model is based on the far-field and first-Born approximations; retarded potential effects are neglected through use of a time-smoothed autocorrelation function of the received electric field signal. A form of the space-time autocorrelation function for electron density fluctuations is derived that accounts for unequal velocity fluctuation components and electron molecular diffusivity to first order. Specific application of the theory is made to axisymmetric, near-wake flows, for which the molecular diffusivity effect on the spectrum function is negligible. In particular, the case of backscattering at small aspect angles is considered. Radial profiles of rms electron density and logitudinal velocity fluctuations, as well as mean velocity, are used that are based on available experimental data. Under the assumption that the spatial electron density spectrum function is constant throughout the scattering volume, relatively simple analytic expressions are arrived at for the first and second moments of the scattered power spectrum function. Numerical results are presented for these moments, interpreted as range rate (or Doppler mean wake velocity) and velocity standard deviation (or Doppler velocity spread). For conditions supported by available data, it is found that, to a first approximation, 1) range rate is equal to the difference between body velocity and mean wake velocity averaged over a wake cross section, U_a, and 2) velocity standard deviation is equal to the difference between the wake front velocity and U_a.

Nomenclature

A = complex amplitude function of received signal
a^* = mean velocity radial distribution parameter
b = sum, Eq. (36)
b_q = electron density and rms velocity fluctuation profile parameters, $q = e,r,\theta,x$
B_q = ratio b_q/a^*
c = in vacuo speed of light
D_e = electron (scalar) molecular diffusivity
f_e,f_i,f_q = dimensionless profile functions, Eqs. (59, 62)
F_i = functions defined by Eqs. (69), $i = 1,2,3,x$
G = general function, Eq. (37)
J_e,J_1,J_2,J_q = profile form factors, Eqs. (60, 62)
$\mathbf{K}$ = scattering vector $\mathbf{k}_i - \mathbf{k}_s$, with components K_j
K = magnitude of $\mathbf{K}$
$\mathbf{k}_i,\mathbf{k}_s$ = incident and scattered wave propagation vectors
k = magnitude of $\mathbf{k}_i$ and $\mathbf{k}_s$
L = axial length of scattering volume
M_0,M_1,M_2 = moments of spectrum function S
$\tilde{n}_e$ = electron density fluctuation
$\hat{n}_e$ = electron density rms fluctuation level
Q = smoothed autocorrelation function of A; displacement probability density
R_e = pure space-autocorrelation function of electron density fluctuations, Eq. (22)
R_f = mean wake radius
$\dot{R}$ = range rate, Eq. (71)
R_λ = turbulent Reynolds number
S = scattered power spectrum function
$\mathcal{S}$ = integral operator, Eq. (26)
S_e = Fourier transform of R_e, Eq. (38)
U_a = area-averaged mean wake velocity, Eq. (74)
U_f = mean velocity at wake edge
$\hat{u}_j$ = rms velocity fluctuation component, $j = 1,2,3$ or $j = r,\theta,x$
$\hat{u}_{j0}$ = component $\hat{u}_j$ on wake axis
$\mathbf{V}$ = mean wake velocity vector, relative to body
W = real part of function A
$\mathbf{x},\mathbf{x}'$ = space coordinate vectors
$\mathbf{Y}$ = displacement vector
Z = imaginary part of A
α,γ = incident and scattered wave aspect angles, measured from flight direction axis
ΔU_0 = mean wake velocity difference, $U_f - \bar{U}_0$
δ_j = jth component of particle displacement
η = dimensionless radius, r/R_f
θ = azimuth angle
κ_q = turbulent Schmidt or kinetic energy number, $q = e,r,\theta,x$
σ_v = velocity standard deviation, Eq. (71)
ϕ = angle between incident and scattered wave planes
$\bar{\omega}$ = frequency shift relative to body frequency shift, Eq. (15)
τ = time lag
ω_0 = carrier frequency
$\Delta\omega$ = standard deviation of frequency shifts, Eq. (13)

Subscripts

0 = axis value, for velocities and electron density
r = radial
x = axial
θ = tangential

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1. Introduction

SCATTERING of electromagnetic waves from weakly ionized, turbulent media has been the subject of theoretical and experimental investigations for a number of years. The over-all objective of these investigations has been to relate statistical properties of the scattered-wave field at a fixed receiver location to the random structural characteristics of the dielectric scattering medium.

Early efforts along these lines were motivated by radar scattering observations from regions of atmospheric turbulence. Extending discrete scattering theory[1] to the case of a continuously distributed field of dielectric fluctuations, Booker and Gordon,[2] and Booker,[3] presented relations for the average radar cross section of a slightly-ionized turbulent region in terms of the electron density fluctuation level and power-spectrum distribution in wave number space. Inclusion of the temporal structure of the scattering medium into the theory was made by Silverman,[4] using a simplified model for the turbulence, which gave a relation between the received signal statistics and the random vertical velocity component in turbulent clouds.

More recently, attention has been given to the problem of electromagnetic scattering from the turbulent, ionized trails produced by bodies entering the atmosphere at hypersonic speeds. Of particular interest in this connection are the mean and fluctuating velocity structure of reentry wakes. Classical Doppler theory says that a discrete scatterer in motion relative to a transmitter radiating coherent waves at a (radian) frequency ω_0 produces a frequency shift from ω_0 in the return signal that is a direct measure of the relative velocity. Since a turbulent wake is a region of continuously distributed dielectric in more-or-less random motion relative to a ground-fixed transmitter, one is thus led by analogy to seek relationships between the scattered power frequency spectrum and the turbulent wake velocity statistics.

The theoretical background for this problem has been presented by Lane.[5] His analysis provides relations between the scattered power frequency spectrum and the velocity structure of the wake for the case of isotropic velocity fluctuations. In essence, his work shows that the mean velocity defect of the wake is primarily responsible for an overall frequency shift in the return signal spectrum when compared to the vehicle frequency shift, whereas the velocity fluctuations are the primary agent for producing a spread in the scattered power spectrum about the frequency at which the peak occurs.

In practice, we are more interested in the frequency content of the return signal as manifested by moments of the spectrum function, not the spectrum function itself. This approach to the turbulence statistics through moments of the resultant scattered power spectrum is quite reasonable, since the spectrum function is validly interpreted as an un-normalized probability distribution of frequencies. At the present time, practical limitations do not warrant consideration of spectrum moments of higher order than the second.

The purpose of this paper is to present general relations for the zeroth, first, and second moments of the scattered power spectrum in terms of the velocity and electron density statistics of the turbulent medium. Extension of previous work in this area will be in two directions. One direction will be in a generalization of the space-time autocorrelation function of electron density fluctuations to the case of nonequal velocity fluctuation components. The second direction will be to use distributions for electron density and velocity fluctuations, as well as mean velocity, which are based on experimental data and are believed to represent improvements on distributions used in previous work.[4,5]

Specific application of the theory will be made to the case of a cylindrical plasma configuration as representative of an axisymmetric, turbulent wake. It will be shown that, in certain situations of practical interest, the final equations for the first and second spectrum function moments are relatively simple, algebraic expressions, even when allowance is made for variations in electron density and velocity fluctuations, and in mean velocity, across the wake.

Part 2 of this paper is devoted to a general discussion of the theory and definitions relevant to the present approach. In Part 3 is derived a space-time correlation function for scalar quantities that applies to the problem treated herein. Part 4 combines the results and definitions of the previous two parts to arrive at the general relations for the first and second moments of the power spectrum and for mean frequency shift and frequency variance. In Part 5 the theory is applied to wake flows, where radial profiles for the relevant turbulent parameters are introduced. Part 6 presents numerical results for the case of backscattering at small aspect angles.

This paper is a shortened version of a report by the present author, Ref. 6. The reader is referred to that report for details of derivations and other information, which have been omitted because of space limitations.

2. Theory

The process under consideration is the scattering of a monochromatic, plane electromagnetic wave from an underdense turbulent plasma. The resultant electric field at a receiver location consists of two parts, 1) a mean component, in general slowly varying with respect to incident signal period, which is caused by scattering from the mean electron density distribution; and 2) a component, which oscillates rapidly compared to the mean, due to scattering from the electron density fluctuations. It is the latter component that contains frequency information about the turbulent field, and to which our attention is directed in what follows.

We denote by $A(t)$ the return signal at a receiver station which corresponds to the electric field component scattered by the dielectric fluctuations. In general, this is a complex quantity, which we can write as

$$A(t) = W(t) + iZ(t) \tag{1}$$

where W and Z are real, continuous functions. A factor $\exp(i\omega_0 t)$ has been suppressed, since this is a modulation at the carrier frequency containing no information about the turbulence.

In practice, we often have at our disposal an ensemble of N sample functions of the process $A(t)$, the mth member of which is $A(t;m)$. This ensemble is generated, for example, by pulsing the transmitted signal. Thus, if the length of individual samples (pulses) is $2T$ sec, each $A(t;m)$ is defined experimentally only within time intervals of $2T$ sec. To analyze the information content of this ensemble in the time (or frequency) domain, we form the ensemble average of lagged products at the time t and $t + \tau$:

$$Q_A(t,t+\tau) = \frac{1}{N}\sum_{m=1}^{N} A(t+\tau;m)A^*(t;m) \equiv \langle A(t+\tau)A^*(t)\rangle \tag{2}$$

where asterisk denotes complex conjugate.

Just what information Q_A contains about the temporal or frequency structure of the turbulence depends, of course, on the relation between A and the scattering dielectric field. In addition, there are many considerations of a practical nature having to do with accuracy of the measurements themselves. Examples of these are 1) frequency resolution of the sampling, $\Delta\omega/2\pi \approx \frac{1}{2}T$, and of the recording equipment, 2) the time span over which the N samples are generated (and thus the number of samples) compared to the time for significant changes in the scattering-field statistics (changes in free-stream conditions, geometrical situation of the body, atmospheric disturbances, etc.), and 3) signal-to-noise levels of the apparatus. A discussion of these aspects of the problem is beyond the scope of this paper, and will not be pursued further (see, for example, Refs. 5, 7–9).

For the present purpose of establishing a theoretical link between the measurable statistics of A and those of the turbulence, we assume that the sample length $2T$ is sufficiently long, and that any other requirements are satisfied, to the extent that Q_A can be written as the two-dimensional Fourier

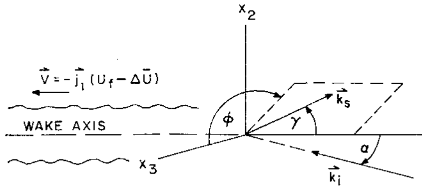

Fig. 1 Geometry of the wave propagation vectors.

transform of a spectrum function $S_A(\omega_1,\omega_2)$[10]

$$Q_A(t,t+\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_A(\omega_1,\omega_2) \times \exp\{i[\omega_2\tau + (\omega_2 - \omega_1)t]\}d\omega_1 d\omega_2 \quad (3)$$

In addition to ensemble averaging, we consider cases wherein a smoothing operation is applied to individual samples. To the same degree of approximation implied by Eq. (3) regarding the sample length $2T$, and since $A(t)$ is continuous so that smoothing and averaging are interchangeable, this can be written formally as

$$Q(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} Q_A(t,t+\tau)dt \quad (4)$$

where $Q(\tau)$ is the smoothed autocorrelation function of the (measured) process. Inserting Eq. (3) into this and reversing the order of integration, one obtains[10]

$$Q(\tau) = \int_{-\infty}^{\infty} S_A(\omega_2,\omega_2)\exp(i\omega_2\tau)d\omega_2 \quad (5)$$

Thus, the smoothed autocorrelation function of the processs $A(t)$ is equal to the Fourier transform of the mass-point distribution of $S_A(\omega_1,\omega_2)$ along the line $\omega_1 = \omega_2$. It is precisely this occupancy that contains all of the frequency information relating to the turbulent field scattering.[5] Therefore, in using the smoothed autocorrelation function, $Q(\tau)$, we need consider only the single-frequency spectrum $S(\omega)$, defined as

$$S(\omega) = S_A(\omega,\omega) \quad (6)$$

Combining Eqs. (2–5) one has

$$Q(\tau) = \langle A(t+\tau)A^*(t)\rangle = \int_{-\infty}^{\infty} S(\omega)\exp(i\omega\tau)d\omega \quad (7)$$

The first equality of this relation is based on the assumption of stationarity of the process $A(t)$. For the pulse lengths and ensemble sample gathering times used in most applications, the statistics of the scattering field are constant in time to a good approximation.

Scattering Model

The basic relationship between the process $A(t)$ and the turbulence is given by the far-field approximation to the retarded-potential solution for the electromagnetic field. Since the retarded potential contributes negligible occupancy along the line $\omega_1 = \omega_2$ in the ω_1,ω_2 plane, we omit retardation effects. Assuming an underdense plasma and single-scattering (first Born approximation), we write[3]

$$A(t) = \int_V \tilde{n}_e(\mathbf{x},t)\exp(i\mathbf{K}\cdot\mathbf{x})d^3\mathbf{x} \quad (8)$$

The coefficient of the integral, which is essentially the Thompson crosssection for an electron, has been set equal to unity, as we will be dealing with normalized moments of the spectrum function; the integration in Eq. (8) is carried out over the volume V of the turbulence.

Our primary interest is evaluation of moments of the spectrum function, so we proceed directly to its general statement via Eq. (7)

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle A(t+\tau)A^*(t)\rangle \exp(-i\omega\tau)d\tau \quad (9)$$

where

$$\langle A(t+\tau)A^*(t)\rangle = \int_V\int_{V'} \langle \tilde{n}_e(\mathbf{x},t)\tilde{n}_e(\mathbf{x},'t+\tau)\rangle \times \exp[i\mathbf{K}\cdot(\mathbf{x}'-\mathbf{x})]d^3\mathbf{x}d^3\mathbf{x}' \quad (10)$$

As found in previous works,[4,5] the space-time autocorrelation function of the electron density fluctuations holds the key to evaluation of $S(\omega)$ and its moments. Before deriving a relation for this function, we present the moment definitions.

Spectrum Function Moments

We note that in the case of a single point-scatterer, the frequency shift in the return signal resulting from the relative motion is unambiguously defined by the classical Doppler shift relation. For the case of scattering from turbulent media, the return signal contains a band of frequencies, and it is not clear a priori that a one-to-one correspondence exists between frequencies in some narrow band $\Delta\omega$ and the velocity amplitude which would produce that shift in the Doppler sense. Neverthless, since measurements[9] show that the scattered power spectrum arises from a relatively narrow-band process, one expects the central tendencies of the spectrum to be closely tied to the velocity statistics of the turbulent field.

In order to define a mean frequency shift and frequency variance (spread) in terms of the velocity field parameters in a body-fixed coordinate system, we first note that the spectrum function $S(\omega)$ is equal in magnitude to the spectrum function of the observed signal, $A(t)\exp(i\omega_0 t)$, but is shifted to the left in frequency space by an amount ω_0. Now the observed spectrum has compact support in the vicinity of ω_0, with moments defined in the frequency half-plane $\omega > 0$. However, because $S(\omega)$ is a shifted spectrum, it can have values occurring in the negative as well as positive frequency domains. Therefore, its moments are defined as

$$M_n = \int_{-\omega_0}^{\infty}\omega^n S(\omega)d\omega \approx \int_{-\infty}^{\infty}\omega^n S(\omega)d\omega (n = 0,1,2) \quad (11)$$

The lower limit $-\omega_0 \to -c/\hat{u}$, which practically is $-\infty$ for all but relativistic fluctuation levels. $S(\omega)$ is not an even function, so all moments are, in general, nonzero. The mean frequency, ω_m, of $S(\omega)$ is thus

$$\omega_m = M_0^{-1}\int_{-\infty}^{\infty}\omega S(\omega)d\omega \equiv M_1/M_0 \quad (12)$$

and the variance, or spread about ω_m, $\Delta\omega^2$, is

$$\Delta\omega^2 = M_0^{-1}\int_{-\infty}^{\infty}(\omega-\omega_m)^2 S(\omega)d\omega \equiv M_2/M_0 - (M_1/M_0)^2 \quad (13)$$

The reentering body produces a frequency shift from ω_0 of magnitude

$$\omega_b = \mathbf{K}\cdot\mathbf{U}_\infty \quad (14)$$

Hence, we can define a mean frequency shift $\bar{\omega}$ as the difference between the frequency shifts produced by the body and wake fluid

$$\bar{\omega} \doteq \omega_b - \omega_m = \mathbf{K}\cdot\mathbf{U}_\infty - M_1/M_0 \quad (15)$$

In effect, we have superimposed a velocity $-\mathbf{j}_1 U_\infty$ (see Fig. 1) onto the actual reentry situation, corresponding now to a coordinate system wherein the body is stationary, with fluid approaching the body at a rate U_∞. Our goal in this analysis is to determine the relations between $\bar{\omega}$, $\Delta\omega$, and the wake turbulence statistics.

3. Space-Time Autocorrelation Function

The work done to date on the space-time autocorrelation function in turbulent flows has been concerned almost solely with the spatio-temporal correlation structure of velocity fluctuations in low-speed grid and boundary-layer flows. In a survey article on the subject, Favre[11] has proposed an analytic form for the autocorrelation coefficient of longitudinal velocity fluctuations, and presents data which give satisfactory support to the theoretical expression for the case of a (homogeneous) grid flow. Lane[5] has used Favre's expression in a somewhat more general form as the basis of an investigation of frequency effects for electromagnetic scattering from turbulent wakes.

We endeavor at this point to derive an expression for the scalar space-time autocorrelation function, appropriate to wake flows, which accounts for unequal velocity components and molecular diffusivity. Let $P(A,B;\mathbf{x}_A,\mathbf{x}_B,\tau)dAdB$ denote the joint probability that some quantity lies between the value A, $A + dA$ at the point $\mathbf{x}_A$ and time t, and between B, $B + dB$ at the point $\mathbf{x}_B$ and time $t + \tau$. Under the assumption that the flow is stationary, the joint probability is a function only of the time difference, τ, as indicated. The space-time correlation function is

$$\langle A(\mathbf{x}_A,t)B(\mathbf{x}_B,t+\tau)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ABP(A,B;\mathbf{x}_A,\mathbf{x}_B,\tau)dAdB \tag{16}$$

Let $Q(\Delta\mathbf{Y};\tau)d^3\Delta\mathbf{Y}$ denote the probability that a fluid element undergoes a displacement ΔY during the interval τ. For a fluid element to arrive at $\mathbf{x}_B$ at the time $t + \tau$, it must be at the point $\mathbf{x}_B - \Delta\mathbf{Y}$ at time t, hence,[12]

$$P(A,B;\mathbf{x}_A,\mathbf{x}_B,\tau) = \int_{-\infty}^{\infty} P(A,B;\mathbf{x}_A,\mathbf{x}_B - \Delta\mathbf{Y},0)Q(\Delta\mathbf{Y};\tau)d^3\Delta\mathbf{Y} \tag{17}$$

Defining a displacement vector $\mathbf{Y}$ such that $\mathbf{x}_B = \mathbf{x}_A + \mathbf{Y} + \Delta\mathbf{Y}$, Eqs. (16) and (17) are combined and the integration order reversed. The double integral over A and B is recognized as the autocorrelation function of A and B corresponding to zero time lag, i.e., the instantaneous spatial correlation. Therefore, one obtains

$$\langle A(\mathbf{x}_A,t)B(\mathbf{x}_B,t+\tau)\rangle = \int_{-\infty}^{\infty} \langle A(\mathbf{x}_A)B(\mathbf{x}_A+\mathbf{Y})\rangle \times Q(\mathbf{x}_B - \mathbf{x}_A - \mathbf{Y};\tau)d^3\mathbf{Y} \tag{18}$$

The crux of the problem lies in choosing a model for the fluid-element displacement probability density appropriate to shear flows. We note that Eq. (18) corresponds formally to the problem of turbulent diffusion from a fixed point source.

Our choice of a displacement model must, of course, be conditioned by the nature of the radar sampling process. In particular, one would expect that the spectrum function moments are governed by the small-time statistical structure of the turbulent field, i.e., the correlation time of the motion itself. Here a distinction must be made between the instantaneous spatial picture of the turbulence seen by the illuminating electromagnetic wave, and the temporal structure that determines frequency content of the return signal. The spatial configuration of the medium is sampled at a wavelength determined by the scattering vector $\mathbf{K}$, whereas the temporal (or frequency) structure within a volume element can contribute a frequency element to the scattered spectrum only during times for which the velocities in any particular direction remain strongly correlated. Thus, for evaluating the part of the return signal that contains the frequency information, it would seem that the relevant time scale associated with the turbulence is the Lagrangian time microscale, λ_t, which is the decorrelation time for directed motion.

Over such short time intervals, the displacement of a fluid particle from its initial position will be small. In addition, the velocity statistics at the point $\mathbf{x}_B$ are related directly (and linearly in first approximation) to those at $\mathbf{x}_A$.[12] To a reasonable approximation, the displacements in the three coordinate directions are independent, being determined only by the velocity component in that direction.

A corresponding simplification, borne out by experiments,[12] is that the displacement probability density is independent of the direction of principal stress axes in the fluid. We thus adopt a displacement model that is formally correct for homogeneous turbulent flows, but that also applies to shear flows for the small time intervals and displacements of concern herein.[12] Thus,

$$Q(\mathbf{x}_B - \mathbf{x}_A - \mathbf{Y};\tau) = [(2\pi)^3\langle\delta_1^2\rangle\langle\delta_2^2\rangle\langle\delta_3^2\rangle]^{-1/2} \times \exp\left[-\frac{1}{2}\sum_1^3 \frac{(x_{jB} - x_{jA} - Y_j - V_j\tau)^2}{\langle\delta_j^2\rangle}\right] \tag{19}$$

where the $\langle\delta_j^2\rangle$, $j = 1,2,3$, are the mean-squared displacements, and V_j is the jth component of mean velocity, which accounts for the gross convection of fluid elements in the absence of fluctuations. In this form, the displacements result from all effects other than mean-velocity convection.

Scalar-Element Displacement Model

The fluid-element displacements δ_j are fundamental measures of the convective-diffusive nature of the turbulent medium. We recall at this point that Eqs. (16–19) are concerned with the motion of fluid elements that are tagged in some sense. If the tagging is just the motion itself, then the displacement model is properly based on turbulent convection only. However, our concern here lies with the space-time structure of the electron density fluctuations. Thus, we interpret the δ_j's as displacements of electron-density elements which are not only subject to convection by the velocity field, but also undergo displacement as a result of molecular diffusion.

For the small diffusion times of interest here, we adopt a linear scalar displacement model, i.e., the total displacement is the sum of the displacements resulting from convection and diffusion[12,13]

$$\langle\delta_j^2\rangle = \langle\tilde{v}_j^2\rangle\tau^2 + 2D_e|\tau| \tag{20}$$

where $\tilde{v}_j$ is the jth component of Lagrangian velocity. The symmetry about $\tau = 0$ is a formal requirement, since $\langle\delta_j^2\rangle$ is always positive.

We note that, according to Eq. (18), the displacement given by Eq. (20) corresponds to fluid elements that were at the point $\mathbf{x}_A + \mathbf{Y}$ at time t and have undergone a displacement δ_j in the interval τ. Since the times are small, we can thus set $\tilde{v}_j$ equal to the Eulerian velocity component $\tilde{u}_j$ at $\mathbf{x}_A + \mathbf{Y}$, i.e.,

$$\langle\tilde{v}_j^2\rangle \approx \langle\tilde{u}_j^2(\mathbf{x}_A+\mathbf{Y})\rangle \equiv \hat{u}_j^2(\mathbf{x}_A+\mathbf{Y}) \tag{21}$$

where the hat denotes rms value.

Application to Electron Density Fluctuations

We now apply the foregoing to the specific case of electron density fluctuations, for which $A = B = \tilde{n}_e$. Defining the spatial autocorrelation R_e as

$$R_e(\mathbf{Y};\mathbf{x}) \doteq \langle\tilde{n}_e(\mathbf{x})\tilde{n}_e\cdot(\mathbf{x}+\mathbf{Y})\rangle[\hat{n}_e(\mathbf{x})\hat{n}_e(\mathbf{x}+\mathbf{Y})]^{-1} \tag{22}$$

and combining this with Eqs. (18) and (19), one has

$$\langle\tilde{n}_e(\mathbf{x},t)\tilde{n}_e(\mathbf{x}',t+\tau)\rangle = \frac{\hat{n}_e(\mathbf{x})}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\frac{\hat{n}_e(\mathbf{x}+\mathbf{Y})R_e(\mathbf{Y};\mathbf{x})d^3\mathbf{Y}}{[\langle\delta_1^2\rangle\langle\delta_2^2\rangle\langle\delta_3^2\rangle]^{1/2}} \times \exp\left\{-\frac{1}{2}\sum_{j=1}^{3}[x_j' - x_j - Y_j - V_j(\mathbf{x}+\mathbf{Y})\tau]^2/\langle\delta_j^2\rangle\right\} \tag{23}$$

where

$$\langle\delta_j^2\rangle = \hat{u}_j^2(\mathbf{x}+\mathbf{Y})\tau^2 + 2D_e(\mathbf{x}+\mathbf{Y})|\tau| \tag{24}$$

as a proposed general form for the space-time autocorrelation function of scalar quantities.

In conclusion of this section, there is one point to be noted. The space-time autocorrelation function given in Eq. (23) is based on convective-diffusive flow models appropriate to small values of the displacement time τ. On the other hand, it is clear from Eq. (9) that the spectrum function $S(\omega)$ consists of contributions from the space-time autocorrelation function for all values of τ from minus infinity to plus infinity. This apparent mismatch is only of a formal nature, however, since the contributions have a relatively abrupt, Gaussian-like cutoff for the times τ which lie beyond the domain of validity of the short-time model. Combination of a long-time displacement model, whereby $\langle \delta_j^2 \rangle \propto t$, with the present model would not affect the final results to any significant degree. Favre's data[11] are strong support for the adequacy of the short-time model used herein.

4. Synthesis of Relations

The foregoing developments will now be combined in order to arrive at the theoretical relations for the scattered power spectrum function and its moments. Substitution of Eq. (23) into Eq. (10) yields[6]

$$Q(\tau) = \langle A(t+\tau)A^*(t)\rangle = \int_V \hat{n}_e(\mathbf{x})d^3\mathbf{x} \int_{-\infty}^{\infty} \hat{n}_e(\mathbf{x}+\mathbf{Y})R_e \times (\mathbf{Y};\mathbf{x})d^3\mathbf{Y}x \exp\left[i\mathbf{K}\cdot(\mathbf{Y}+\mathbf{V}\tau) - \frac{1}{2}\tau^2\Sigma(K_j\hat{u}_j)^2 - D_eK^2|\tau|\right] \quad (25)$$

for the smoothed autocorrelation function of the turbulent electric field component.

As a temporary convenience in writing the moment relations, let us define the integral operator $\mathcal{S}$ as

$$\mathcal{S} \doteq \int_V \hat{n}_e(\mathbf{x})d^3\mathbf{x} \int_{-\infty}^{\infty} \hat{n}_e(\mathbf{x}+\mathbf{Y})R_e(\mathbf{Y};\mathbf{x}) \exp(i\mathbf{K}\cdot\mathbf{Y})d^3\mathbf{Y} \quad (26)$$

Comparing this with Eq. (25), one sees that $\mathcal{S}$ operates on the time-dependent part of the integrand. It follows from Eq. (9) that the scattered power spectrum function is given by

$$S(\omega) = \frac{1}{\pi}\mathcal{S}\int_0^\infty \cos[(\omega - \mathbf{K}\cdot\mathbf{V})\tau] \times \exp\left[-\frac{1}{2}\tau^2\sum_{j=1}^{3}(K_j\hat{u}_j)^2 - D_eK^2\tau\right]d\tau \quad (27)$$

in its most general form, according to the present model.

Neglect of Molecular Diffusivity D_e

From the form of the integrand in Eq. (27), it is highly desirable to neglect the molecular diffusivity term in the exponent so as to simplify the moment relations. Clearly, the effect of D_e can be neglected when the exponential fall-off is dominated by the quadratic term, i.e., when

$$\beta \doteq D_eK^2[2/\Sigma(K_j\hat{u}_j)^2]^{1/2} \ll 1 \quad (28)$$

For order-of-magnitude considerations, let us assume an isotropic turbulence, for which $\hat{u}_1 = \hat{u}_2 = \hat{u}_3 = \hat{u}$, as well as the case of backscattering, whereby $K = 4\pi/\lambda_R$, where λ_R is the radar wavelength. Then Eq. (28) becomes

$$\beta = \frac{4\pi 2^{1/2}D_e}{\hat{u}\lambda_R} = \frac{4\pi 2^{1/2}}{(\nu/D_e)R_\lambda}\left(\frac{\lambda_g/L_f}{\lambda_R/L_f}\right) \ll 1 \quad (29)$$

where ν = kinematic viscosity of the fluid, λ_g and L_f are the Taylor microscale and integral scale of the turbulence, and $R_\lambda \doteq \hat{u}\lambda_g/\nu$ = turbulent Reynolds number. Now using Batchelor's and Taylor's relations for the dissipation rate[14,15]

$$\Phi \approx \hat{u}^3/L_f \text{ and } \Phi = 15\nu(\hat{u}/\lambda_g)^2 \quad (30)$$

respectively, to write λ_g/L_f in terms of R_λ, Eq. (29) becomes

$$\beta \approx (270/R_\lambda^2)(D_eL_f/\nu\lambda_R) \ll 1 \quad (31)$$

According to this criterion, there is no general justification for neglecting the effect of electron molecular diffusivity on the scattered power spectrum function. In the relatively near-wake region behind bodies in hypersonic flight, the electron diffusion is ambipolar, and D_e is not greater than ν by more than a factor of about 4. Since λ_R is of the order of L_f, and R_λ is expected to be $\gtrsim 100$, the criterion $\beta \ll 1$ is satisfied in these situations. In other cases, such as free diffusion of electrons resulting from relatively high negative ion concentrations, D_e can be $\gg \nu$, so that $\beta \ll 1$ requires very large R_λ and/or $L_f \ll \lambda_R$.

For the purposes of this paper, it is assumed that $\beta \ll 1$. Therefore, to the relations to be presented subsequently, we attach the caveat that their use, in any given situation (other than our application to near-wake flows), be justified by careful evaluation of the criterion, Eq. (28).

Relations for the Case of Negligible Diffusivity

Setting $D_e = 0$ and evaluating the integral in Eq. (27) yields

$$S(\omega) = \mathcal{S}[2\pi\Sigma(K_j\hat{u}_j)^2]^{1/2} \exp\left[-\frac{1}{2}(\omega - \mathbf{K}\cdot\mathbf{V})^2/\Sigma(K_j\hat{u}_j)^2\right] \quad (32)$$

as a generalized form of the scattered power spectrum function. The frequency dependence is Gaussian, but weighted by the velocity and electron density fluctuations, and mean velocity distributions.

Evaluation of the moments from Eq. (32) is straightforward. One obtains

$$M_0 = \mathcal{S} \quad (33)$$

$$M_1 = \mathcal{S}\mathbf{K}\cdot\mathbf{V} \quad (34)$$

$$M_2 = \mathcal{S}[b + (\mathbf{K}\cdot\mathbf{V})^2] \quad (35)$$

where

$$b \equiv \sum_{j=1}^{3}(K_j\hat{u}_j)^2 \quad (36)$$

as generalized relations for the zeroth, first, and second spectrum function moments neglecting molecular diffusivity effects.

Approximate Expressions

The moment relations given above are tedious evaluate for most situations of interest. In order to simplify these equations, we note that both $\mathbf{V}$ and b are functions only of the position vector $\mathbf{x} + \mathbf{Y}$. Hence, recalling the definition of the integral operator $\mathcal{S}$, Eq. (29) all terms in Eqs. (33–36) are of the form

$$G \doteq \int_V \hat{n}_e(\mathbf{x})d^3\mathbf{x}\int_{-\infty}^{\infty} F(\mathbf{x}+\mathbf{Y})R_e(\mathbf{Y};\mathbf{x}) \exp(i\mathbf{K}\cdot\mathbf{Y})d^3\mathbf{Y} \quad (37)$$

where F denotes a general function of $\mathbf{V}$ and $\hat{u}_j$ (or b) as determined from Eqs. (33–36). Recognizing that, apart from F, the integral over $\mathbf{Y}$ in Eq. (37) is just the spectrum function of the electron density fluctuations evaluated at $\mathbf{K}$, i.e.,

$$S_e(\mathbf{K};\mathbf{x}) \doteq \int_{-\infty}^{\infty} R_e(\mathbf{Y};\mathbf{x}) \exp(i\mathbf{K}\cdot\mathbf{Y})d^3\mathbf{Y} \quad (38)$$

it is clear that a great deal of work would be saved if one could write

$$G \approx \int_V \hat{n}_e(\mathbf{x})F(\mathbf{x})S_e(\mathbf{K};\mathbf{x})d^3\mathbf{x} \quad (39)$$

Approximate conditions under which Eq. (39) is valid are established in Ref. 6. Under the assumption that R_e has a Gaussian form

$$R_e(\mathbf{Y};\mathbf{x}) = \exp[-(Y/\lambda_e)^2] \quad (40)$$

corresponding to an isotropic turbulence, Eq. (39) is valid if

$$\tfrac{1}{2}(K_j\lambda_e)[(\lambda_e/F)(\partial F/\partial x_j)]_{\mathbf{x}} \ll 1 \tag{41}$$

for all j.

No statement can be made regarding how well Eq. (41) is satisfied in general; each flow configuration must be examined separately. Since our interest lies mainly in applying the present theory to wake flows, we consider the case of backscattering from wakes at small aspect angles. In this case, the scattering vector K is directed along, or nearly so, the wake axis. Then $K_2 = K_3 \approx 0$, and, with $|\mathbf{K}| = 4\pi/\lambda_R$, Eq. (41) is

$$2\pi(\lambda_e/\lambda_R)[(\lambda_e/F)(\partial F/\partial x_1)]_{\mathbf{x}} \ll 1 \tag{42}$$

where $\mathbf{x}_1$ is in the axis direction. Now the decorrelation distance of the electron density fluctuations is of the order of the mean wake radius. Over this distance, the fractional change of $\mathbf{V}$, $\hat{n}_e$, and $\hat{u}_1$ in the axial direction, at any position within the wake, is extremely small. Hence, the fractional change in F is correspondingly minute, and Eq. (42) is satisfied for any realistic values of λ_e/λ_R.

In light of these considerations, we assume in what follows that Eq. (42) is satisfied, i.e., that Eq. (39) is a valid approximation to Eq. (37). Therefore, the operator $\mathcal{S}$ defined in Eq. (26) is, for all subsequent relations

$$\mathcal{S} = \int_V \hat{n}_e^2(\mathbf{x})S_e(\mathbf{K};\mathbf{x})d^3\mathbf{x} \tag{43}$$

Consistent with this, $\mathbf{V}$ and b are functions only of the position vector $\mathbf{x}$. Therefore, reverting to the full notation

$$M_0 = \int_V \hat{n}_e^2(\mathbf{x})S_e(\mathbf{K};\mathbf{x})d^3\mathbf{x} \tag{44}$$

$$M_1 = \int_V \hat{n}_e^2(\mathbf{x})S_e(\mathbf{K};\mathbf{x})\{\mathbf{K}\cdot\mathbf{V}(\mathbf{x})\}d^3x \tag{45}$$

$$M_2 = \int_V \hat{n}_e^2(\mathbf{x})S_e(\mathbf{K};\mathbf{x})\left\{[\mathbf{K}\cdot\mathbf{V}(\mathbf{x})]^2 + \sum_1^3 [K_j\hat{u}_j(\mathbf{x})]^2\right\} \tag{46}$$

If $\mathbf{V}$ and $\hat{u}_j$ are constant throughout the scattering volume, one obtains

$$\bar{\omega} = \mathbf{K}\cdot(\mathbf{U}_\infty - \mathbf{V}) \quad \text{and} \quad \Delta\omega^2 = \Sigma(K_j\hat{u}_j)^2 \tag{47}$$

for the mean frequency shift and frequency variance. This result confirms directly our earlier intuitive feeling about the connection between Doppler observables and the velocity field.

5. Application of Theory to Wake Flows

The foregoing theoretical developments are applied in this section to scattering from axisymmetric, cylindrical plasmas that have relatively slow variations in velocity and electron density statistics in the axial direction. To a first approximation, this is the situation obtaining in hypersonic wake flows.

Geometrical Considerations

The geometry of the scattering vectors is shown in Fig. 1. Since the flow is axisymmetric, one can always define a plane of incidence that contains the vehicle velocity vector, $\mathbf{j}_1U_\infty$, and the incident-wave propagation vector, $\mathbf{k}_i$. This is the x_1,x_3 plane. The scattered-wave vector, $\mathbf{k}_s$, lies in a plane at angle ϕ to the plane of incidence; α and γ denote the angles of the incident and scattered-wave vectors with respect to the vehicle velocity vector. In terms of these angles, one has

$$\mathbf{K} = \mathbf{k}_i - \mathbf{k}_s = -k[\mathbf{j}_1(\cos\gamma + \cos\alpha) + \mathbf{j}_2\sin\phi\sin\gamma + \mathbf{j}_3(\cos\phi\sin\gamma + \sin\alpha)] \tag{48}$$

where the $\mathbf{j}_n$ are unit vectors in the (fixed) x_1, x_2, and x_3 direction. Angle α is the aspect, or look, angle. Note that for backscattering, $\gamma = \alpha$ and $\phi = 0$.

We consider in this paper only in-plane scattering, with $\phi = 0$. Thus, for present purposes, there is no component of $\mathbf{K}$ out of the incidence plane, i.e., $K_2 = 0$. It follows that the magnitude of $\mathbf{K}$ is

$$|\mathbf{K}| \equiv K = k\{2[1 + \cos(\alpha - \gamma)]\}^{1/2} \tag{49}$$

The mean velocity is assumed to be purely axial, which in the present coordinate system moving with the body is

$$\mathbf{V} = \mathbf{j}_1(\bar{U} - U_\infty) = -\mathbf{j}_1(U_\infty - U_f + \Delta\bar{U}) \tag{50}$$

where $\Delta\bar{U}$ is the mean wake velocity defect relative to the front velocity U_f. Using Eq. (48), then, one has

$$\mathbf{K}\cdot\mathbf{V} = k(\cos\gamma + \cos\alpha)(U_\infty - U_f + \Delta\bar{U}) \tag{51}$$

in general. For the present cylindrical geometry, a volume element is

$$d^3\mathbf{x} = r\,dr\,d\theta\,dx \tag{52}$$

where θ is the elevation angle of $d^3\mathbf{x}$ with respect to the plane of incidence.

In the case of axial symmetry, the appropriate rms velocity fluctuation components are in the axial, radial, and tangential directions, which are denoted by $\hat{u}_x$, $\hat{u}_r$, and $\hat{u}_\theta$, respectively. The Cartesian x_3 component is

$$\hat{u}_3 = \hat{u}_r\cos\theta - \hat{u}_\theta\sin\theta \tag{53}$$

$\hat{u}_x$, $\hat{u}_r$, and $\hat{u}_\theta$ are functions only of x and r by assumption. Defining the angle functions

$$g_s = \sin\gamma + \sin\alpha \quad g_c = \cos\gamma + \cos\alpha \tag{54}$$

and using Eq. (48) one has

$$\Sigma(K_j\hat{u}_j)^2 = k^2\{(g_c\hat{u}_x)^2 + [g_s(\hat{u}_r\cos\theta - \hat{u}_\theta\sin\theta)]^2\} \tag{55}$$

As a reasonably accurate description of the turbulent property distributions throughout the scattering volume, we make the assumption that the radial variations are locally similar. For the similarity variable in the radial direction, η, we use the physical radius r divided by the mean wake radius, R_f, whereby†

$$\eta = r/R_f(x)$$

Thus, any variable $z(\mathbf{x})$, say, can be written as

$$z(\mathbf{x}) = z(x,r) = z_0(x)[z(x,r)/z_0(x)] \equiv z_0(x)f_z(\eta) \tag{56}$$

where z_0 is the value of z on the wake axis and f_z is a dimensionless function of η only. Integrals of the various f_z's over a wake cross section will thus represent form factors, which will be evaluated later on.

Moment Relations for Wake Flows

We now proceed to evaluate Eqs. (45–47) for wake flows. A simplifying assumption, which is made in order to arrive at concrete results, is that the electron density spectrum function, S_e, is constant over any cross section, but may be a function of axial distance x, i.e.,

$$S_e(\mathbf{K};\mathbf{x}) = S_e(\mathbf{K};x) \tag{57}$$

The zeroth moment, given by Eq. (44) can thus be written as

$$M_0 = 2\pi\int_0^L \hat{n}_{e_0}^2(x)R_f^2(x)S_e(\mathbf{K};x)dx\int_0^\infty f_e(\eta)\eta d\eta \tag{58}$$

where

$$f_e(\eta) \doteq [\hat{n}_e(\mathbf{x})/\hat{n}_{e_0}]^2 \equiv [\hat{n}_e(x,r)/\hat{n}_e(x,0)]^2 \tag{59}$$

† R_f is defined as the (physical) radius at which equal probability of turbulent and nonturbulent flow obtains, i.e., the radius at which the intermittency factor $= \frac{1}{2}$.

The length L represents the axial extent of the scattering volume over which we are performing the integration. For our purposes, L is arbitrary, and can be made small enough so that the quantities $\hat{n}_{e_0}$, R_f, etc. are approximately constant. Hence, denoting the integral of f_e by J_e

$$J_e = \int_0^\infty f_e(\eta)\eta d\eta \tag{60}$$

Eq. (58) becomes

$$M_0 = 2\pi J_e(\hat{n}_{e_0}R_f)^2 S_e(\mathbf{K})L \tag{61}$$

where x-dependence has been suppressed.

Similar treatment can be given to Eqs. (45) and (46) for M_1 and M_2. With the following definitions

$$f_u(\eta) = \Delta\bar{U}/\Delta U_0 \qquad J_1 = \int_0^\infty f_e(\eta)f_u(\eta)\eta d\eta$$

$$J_2 = \int_0^\infty f_e(\eta)f_u{}^2(\eta)\eta d\eta \tag{62}$$

$$f_q(\eta) = [\hat{u}_q(\mathbf{x})/\hat{u}_{q_0}]^2 \qquad J_q = \int_0^\infty f_e(\eta)f_q(\eta)\eta d\eta$$

where $q = x, r$, and θ, one obtains for the mean frequency shift and frequency variance given by Eqs. (15) and (13)[6]

$$(\bar{\omega}/k) = [U_\infty - U_f + (J_1/J_e)\Delta U_0](\cos\alpha + \cos\gamma) \tag{63}$$

$$\left(\frac{\Delta\omega}{k}\right)^2 = \left\{\left[\frac{J_2}{J_e} - \left(\frac{J_1}{J_e}\right)^2\right]\Delta U_0{}^2 + \frac{J_x}{J_e}\hat{u}_{x_0}{}^2\right\} \times$$
$$(\cos\alpha + \cos\gamma)^2 + \frac{1}{2}\left(\frac{J_r}{J_e}\hat{u}_{r_0}{}^2 + \frac{J_\theta}{J_e}\hat{u}_{\theta_0}{}^2\right)(\sin\alpha + \sin\gamma)^2 \tag{64}$$

From Eq. (63), the mean frequency shift consists of two terms, one term, $\propto U_\infty - U_f$, representing a pure translation of the turbulent core at the local front velocity, and the other term, $\propto \Delta U_0$, representing the weighted translation of the turbulent fluid itself. The frequency variance is seen to be governed primarily by the velocity fluctuations, but also contains a contribution from the mean-velocity field, which in itself is a sort of large-scale fluctuation.

Radial Profiles

In order to arrive at numerical values of the mean frequency shift and frequency standard deviation, we utilize the following radial profile models for the turbulent wake properties.

Mean wake velocity

The radial profile used for the mean wake velocity is a Gaussian distribution in physical coordinates:

$$(U_f - \bar{U})/(U_f - \bar{U}_0) \equiv \Delta\bar{U}/\Delta U_0 = \exp(-a^*\eta^2) \equiv f_u(\eta) \tag{65}$$

The form of Eq. (65) is based on results from low-speed,[16] supersonic,[17] and hypersonic[18] wake experiments. Further support for this mean-velocity distribution is obtained from data on heated jets,[19,20] for which the corresponding incompressible mean-velocity profile is also Gaussian or very nearly so.[21] (Replotting the data of Ref. 20 in physical coordinates gives as good a fit to a Gaussian curve as the data plotted as a function of the "compressible" radial coordinate, if not a slightly better fit.)

Electron density fluctuation

The transverse distribution of passive scalar fluctuations in free turbulent flows[22,23] contains an off-axis peak resulting from local production, and is accurately described by the form

$$\hat{n}_e(x,r)/\hat{n}_{e0}(x) = (1 + b_e\eta^2)\exp(-a^*\kappa_e\eta^2) \equiv f_e{}^{1/2} \tag{66}$$

The parameter κ_e is basically the electron density turbulent Schmidt number, which is defined as the width of the mean electron density profile compared to the mean velocity profile, i.e.,[21]

$$\bar{n}_e(x,r)/\bar{n}_{e_0}(x) = (\Delta\bar{U}/\Delta U_0)^{\kappa_e} \tag{67}$$

Strictly speaking, the mean electron concentration, or mass fraction, should appear in Eq. (67) rather than the electron density, since both theory[21] and experiments[19] indicate that the concentration has a self-preserving distribution. However, as shown in Ref. 24, an equivalent κ_e can be defined for a Gaussian distribution of electron density, which gives a reasonably accurate profile if one accounts properly for the radial variation in gas density.

Velocity fluctuations

Data on the three rms velocity components in fully-developed turbulent wakes is limited to incompressible flows.[16] The components $\hat{u}_1$ and $\hat{u}_3$ have an off-axis peak, probably as a result of $\hat{u}_1$ production and energy transfer from the $\hat{u}_1$ to the $\hat{u}_3$ component, whereas $\hat{u}_2$ seems to have a more nearly Gaussian distribution, which may reflect the boundary (wake-edge) constraints in the transverse direction. To account for these distributions, the form

$$\hat{u}_q/\hat{u}_{q_0} = (1 + b_q\eta^2)\exp(-a^*\kappa_q\eta^2) \equiv f_q{}^{1/2} \tag{68}$$

is adopted, where q may be x, r, or θ.

As with the electron density distribution, the parameter κ_q is a turbulent component kinetic-energy number, which accounts for the distribution of $\hat{u}_q$ being wider or narrower than the mean-velocity profile. Also, the parameter b_q determines the peak amplitude and position of the $\hat{u}_q$ profile.

The profile models given by Eqs. (65), (66), and (68) are a parametric representation of the wake turbulence structure based on available experimental data. The hope is, of course, that these parameters are "universal," or nearly so, and that their values can be obtained from existing data within acceptable bounds. This seems to be the case for the velocity distribution parameters a^*, b_x, and κ_x, and to a lesser extent for b_r, b_θ, κ_r, and κ_θ. On the other hand, the electrons can undergo complex chemistry in wakes,[25] which could affect the values of b_e and κ_e, making these parameters vary as functions of x in a nonnegligible manner. Much work remains to be done in this area. The sensitivity of mean frequency shift and frequency standard deviation to these parameters will be seen shortly.

Radial Profile Integrals

Using the foregoing profile models, one can obtain explicit relations for the profile integrals and their ratios. To this end, we define the following quantities.

$$F_1 = 1 + B_e/\kappa_e\,(1 + B_e/2\kappa_e) \tag{69a}$$

$$F_2 = 1 + 2B_2/(1 + 2\kappa_e)[1 + B_e(1 + 2\kappa_e)] \tag{69b}$$

$$F_3 = 1 + B_e/(1 + \kappa_e)[1 + (B_e/2)/(1 + \kappa_e)] \tag{69c}$$

$$F_q = 1 + \frac{B_e + B_q}{\kappa_e + \kappa_q}\left[1 + \frac{B_e + B_q}{2(\kappa_e + \kappa_q)}\right] +$$
$$\frac{B_eB_q}{(\kappa_e + \kappa_q)^2}\left[1 + \frac{3/2}{\kappa_e + \kappa_q}\left(\frac{B_e + B_q}{2} + \frac{B_eB_q}{\kappa_e + \kappa_q}\right)\right] \tag{69d}$$

The integral ratios can thus be written as

$$J_1/J_e = [2\kappa_e/(1 + 2\kappa_e)](F_2/F_1) \tag{70a}$$

$$J_2/J_e = [\kappa_e/(1 + \kappa_e)](F_3/F_1) \tag{70b}$$

$$J_q/J_e = [\kappa_e/(\kappa_e + \kappa_q)][F_q/F_1) \tag{70c}$$

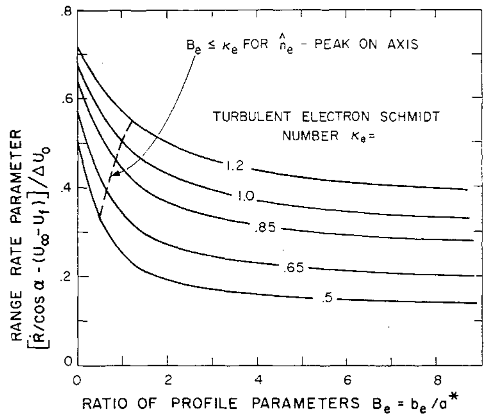

Fig. 2 Range rate parameter vs amplitude parameter of the electron density fluctuation profile. Case for backscattering. The values of electron turbulent Schmidt number κ_e encompass the range as given in the appendix of Ref. 6.

6. Results for Backscattering

To illustrate the preceding theoretical developments, we consider the case of backscattering at small aspect angles. For this situation, we have $\gamma = \alpha$ (see Fig. 1). We restrict the results to aspect angles that are small enough to allow neglect of the second term on the right in Eq. (64), i.e.,

$$\tan^2\alpha \ll 2J_x\hat{u}_{x0}^2/(J_r\hat{u}_{r0}^2 + J_\theta\hat{u}_{\theta0}^2) \sim 1$$

An upper bound on α for these results would thus be in the neighborhood of 30°.

Since the left-hand sides of Eqs. (63) and (64) have the dimensions of velocity and velocity squared, it is common practice to use the appellations range rate, or mean-wake velocity, associated with the mean frequency shift, and velocity standard deviation, or velocity spread, associated with the (square root of) frequency variance. By analogy with Eq. (47), the magnitude of the scattering vector, K, is used as the inverse length scale. Hence, we have the following definitions

$$\text{range rate } \dot{R} \doteq \bar{\omega}/K = \bar{\omega}/2k \quad (71)$$

$$\text{velocity standard deviation } \sigma_v \doteq \Delta\omega/2k$$

where the denominator $2k = |\mathbf{K}|$ by Eq. (49) for backscattering. Equations (63) and (64) can thus be written as, using Eq. (70),

$$\dot{R}/\cos\alpha = U_\infty - U_f + 2\kappa_e/(1 + 2\kappa_e)(F_2/F_1)\Delta U_0 \quad (72)$$

$$\frac{\sigma_v}{\cos\alpha} = \left\{\left[\frac{F_1F_3\kappa_e}{1+\kappa_e} - \left(\frac{2F_2\kappa_e}{1+2\kappa_e}\right)^2\right]\left(\frac{\Delta U_0}{F_1}\right)^2 + \frac{\kappa_e}{\kappa_e+\kappa_x}\left(\frac{F_x}{F_1}\right)\hat{u}_{x0}^2\right\}^{1/2} \quad (73)$$

for backscattering at small aspect angles.

Range Rate $\dot{R}$

The range rate expression is presented in a general form by defining a range rate parameter

$$[\dot{R}/\cos\alpha - (U_\infty - U_f)]/\Delta U_0$$

which is a function only of the electron turbulent Schmidt number κ_e and amplitude parameter $B_e = b_e/a^*$. This is shown in Fig. 2.

The behavior exhibited in Fig. 2 can be understood by considering the weighting of the mean velocity field by the electron fluctuation profile. Consider a fixed value of κ_e, along with a fixed mean-velocity (defect) profile, i.e., a^* = constant. The weighting now occurs only through the radial distribution of $\hat{n}_e$. For $b_e \le \kappa_e$, the $\hat{n}_e$ profile has its peak on the axis, which is clearly a weighting of the high velocity-defect region near the axis; the maximum possible weighting occurs for $b_e = 0$. As the $\hat{n}_e$ peak moves off the axis, lower velocity-defect regions are increasingly emphasized by the broader electron fluctuation profile. This emphasis of lower velocity-defect regions is limited, of course, by the $\Delta\bar{U}$ profile itself (we are integrating $\hat{n}_e^2\Delta\bar{U}$), i.e., when $B_e \gtrsim 2$, the $\hat{n}_e$ peak occurs at radii for which $\Delta\bar{U} \to 0$. For $B_e \gg 1$, both F_1 and F_2 are proportional to B_e^2 as a result of the increasing amplitude of the $\hat{n}_e$ profile.

Instead of using the total wake velocity defect, ΔU_0, one can normalize the range rate by the area-average velocity defect, $U_f - U_a$, where

$$U_a \doteq (2/R_f^2)\int_0^{R_f}\bar{U}rdr \quad (74)$$

is the mean velocity averaged over a wake cross section. Using Eq. (65) one has

$$U_f - U_a = \Delta U_0/a^* \quad (75)$$

where a term exp $(-a^*)$ has been neglected compared to unity. Accordingly, a modified range rate parameter is defined as

$$[\dot{R}/\cos\alpha - (U_\infty - U_f)]/(U_f - U)$$

This is shown in Fig. 3 as a function of a^* for various values of the parameter B_e. The data area indicated in the figure, along with the value $\kappa_e = 1$ used to calculate the curves, is discussed in the Appendix of Ref. 6.

From the figure, a reasonable approximate value for the modified range rate parameter is seen to be unity, or,

$$\dot{R}/\cos\alpha \approx U_\infty - U_a \quad (76)$$

In other words, the mean frequency shift is, to a first approximation, a direct measure of the area-averaged mean wake velocity.

Velocity Standard Deviation σ_v

Since Fig. 2 gives a complete picture of values for the range rate parameter, we present results for the velocity standard deviation as a ratio of $\sigma_v/\cos\alpha$ to the reduced range rate

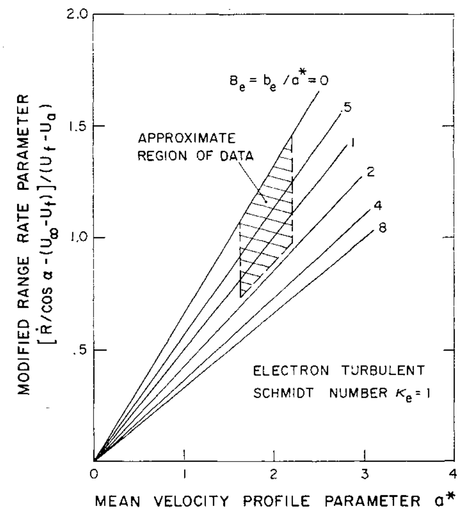

Fig. 3 Modified range rate vs mean-velocity profile parameters for backscattering. The velocity U_a is a cross-section averaged value. Parameter values delineating the data region are given in the appendix of Ref. 6.

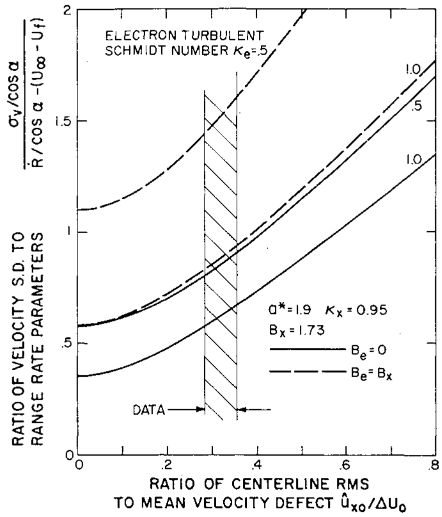

Fig. 4 Ratio of velocity standard deviation (S.D.) to "reduced" range rate as a function of rms fluctuation-to-mean velocity ratio; values of the profile parameters used in the plot are discussed in the appendix of Ref. 6.

$\dot{R}/\cos\alpha - (U_\infty - U_f)$. From Eqs. (72) and (73) one has

$$\frac{\sigma_v/\cos\alpha}{\dot{R}/\cos\alpha - (U_\infty - U_f)} = \frac{1 + 2\kappa_e}{2F_2}\left[\frac{F_1F_3/\kappa_e}{1 + \kappa_e} - \left(\frac{2F_2}{1 + 2\kappa_e}\right)^2 + \frac{F_1F_x/\kappa_e}{\kappa_e + \kappa_x}\left(\frac{\hat{U}_{x0}}{\Delta U_0}\right)^2\right]^{1/2} \quad (77)$$

This is presented in Fig. 4 for the value of $\kappa_e = 0.5$ and 1.0, along with $B_e = 0$ and B_x; the data band in the figure corresponds to experimental data on $\hat{u}_{x0}/\Delta U_0$, as discussed in the Appendix of Ref. 6.

Although there is a somewhat larger range of values for this ratio compared to the modified range rate parameter, a representative value is seen to be about unity within the given data band. Therefore, as a first approximation, we take

$$\sigma_v/\cos\alpha \approx \dot{R}/\cos\alpha - (U_\infty - U_f)$$

or, using Eq. (76),

$$\sigma_v/\cos\alpha \approx U_f - U_a \quad (78)$$

It follows that

$$\sigma_v/\dot{R} \approx (U_f - U_a)/(U_\infty - U_a) \quad (79)$$

for backscattering at small aspect angles.

The physical basis for Eq. (78) is that the frequency spread about the peak is governed mainly by the velocity fluctuations; these in turn are driven by the local difference in mean velocity across the turbulent zone. Finally, we note that the ratio $\sigma_v/\dot{R}$ given by Eq. (79) is less than unity in the near-wake region by an amount that depends upon the difference between U_∞ and U_f.

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